On the Fourier transform of temperate weight function k_s and the norm in space $\mathscr{H}_{(s)}$

by

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Introduction. We know that it is important to give precise descriptions concerning the regularity of the solutions for partial differential equations. For example, the theory on convolution of function with the Poisson kernel or the Gauss-Weierstrass kernel and on its Fourier transform is known [1]. The Poisson kernel and the Gauss-Weierstrass kernel on $R_{n+1,+}$* are such functions

$$P(x, y)=\pi^{-\frac{n+1}{2}}\Gamma(\frac{n+1}{2})\frac{y}{(|x|^2+y^2)^{\frac{n+1}{2}}}$$

$$W(x, y)=2^{-n}\pi^{-\frac{n}{2}}y^{-\frac{n}{2}}\exp(\frac{-|x|^2}{4y})$$

and their Fourier transforms are $\hat{P}(\xi, y)=\exp(-2\pi y|\xi|)$ and $\hat{W}(\xi, y)\doteq\exp(-4\pi^2 y|\xi|^2)$ for elements x, ξ of the real n-dimensional space R_n and positive y.

Moreover it is described that a condition on the regularity of a distribution or function u (with compact support) can be regarded as a condition on the behavior at infinity of the Fourier transform $\hat{u}$ [2]. According to Hörmander's idea, a way to classify this behavior is to ask for which weight functions k, $k\cdot\hat{u}\in L_p$ is true ([2], [3]). Especially, of the case when k is a temperate weight function k_s ;

$$k_s(\xi)=(1+|\xi|^2)^{s/2},$$

fruitful studies have been done ([1], [2], [3], [4]).

Now, we denote by $\mathscr{H}_{(s)}$ the set of all distributions $u\in\mathscr{S}'$** such that $\hat{u}$ is a function and

$$\|u\|_{(s)}=(\int_{R_n}|\hat{u}(\xi)\cdot k_s(\xi)|^2\,d\xi)^{1/2}<\infty.$$

This is a Banach space with norm $\|u\|_{(s)}$ and the space $C_0^\infty(R_n)$ is dense in $\mathscr{H}_{(s)}$ [2].

This note contains the calculation of elementary formula which we need in our another work aiming to self-contain consepts. In § 1 we describe the Bochner's theorem with its proof and in § 2 we derive the Fourier transform of k_s that is different to the known formula at coefficient, from this theorem according to Aronszajn's method. In § 3 we shall discuss a relation of the norm on $\mathscr{H}_{(s)}$ and the L_2 norm of regularization *** of u.

§ 1. We defined above the Fourier transform $\hat{F}$ of $F(x, y)$ for $x\in R_n$, $y>0$ by

$$\hat{F}(\xi, y)=\int_{R_n}F(x, y)\exp(-2\pi i\xi\cdot x)\,d\chi.$$

Evidently $\hat{\hat{F}}=\check{F}$ which is $F(-x, y)$ and y disappears in $\hat{f}$ for $f(x)$.

Theorem (Bochner). If the integrable function $f(x)$ depends on absolute value x only then $\hat{f}(\xi)$ is also the same type.

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Namely, if $f(x_1, x_2, \cdots, x_n)=\varphi(\sqrt{x_1^2+x_2^2\cdots+x_n^2})$ then

$$(1)\qquad \hat{f}(\xi_1, \xi_2, \cdots, \xi_n)=(2\pi)^{-\frac{n}{2}}|\xi|^{-\frac{n}{2}+1}\int_0^\infty \varphi(\frac{\rho}{2\pi})\rho^{\frac{n}{2}}J_{\frac{n-2}{2}}(|\xi|\rho)d\rho$$

where J_λ is the Bessel function of first kind. Especially, when we take $|\xi|^2=t$

$$(2)\qquad \hat{f}(\xi_1, \xi_2, \cdots, \xi_n)=(-1)^{\frac{n}{2}-1}2^{-1}\pi^{-\frac{n}{2}}\frac{d^{(\frac{n-2}{2})}}{dt^{(\frac{n-2}{2})}}\int_0^\infty \varphi(\frac{\rho}{2\pi})\rho\, J_0(\sqrt{t}\rho)d\rho$$

for even n and

$$(3)\qquad \hat{f}(\xi_1, \xi_2, \cdots, \xi_n)=(-1)^{\frac{n-1}{2}}2^{-\frac{1}{2}}\pi^{-\frac{n}{2}}\frac{d^{(\frac{n-1}{2})}}{dt^{(\frac{n-1}{2})}}\left\{t^{\frac{1}{4}}\int_0^\infty \varphi(\frac{\rho}{2\pi})\rho^{\frac{1}{2}}J_{-\frac{1}{2}}(\sqrt{t}\rho)d\rho\right\}$$

for odd n.

Proof. For the completion we show the following known formulas [6] with their proofs at first;

$$(4)\qquad J_\lambda(z)=\frac{1}{\Gamma(\frac{1}{2})\Gamma(\lambda+\frac{1}{2})}(\frac{z}{2})^\lambda\int_{-1}^1(1-t^2)^{\lambda-\frac{1}{2}}\exp(izt)dt$$

$$=\frac{\Gamma(\frac{1}{2}-\lambda)}{2\pi i\Gamma(\frac{1}{2})}(\frac{z}{2})^\lambda\int_{C_1}(\zeta^2-1)^{\lambda-\frac{1}{2}}\exp(iz\zeta)d\zeta$$

$$(5)\qquad J_{-\lambda}(z)=\frac{\Gamma(\frac{1}{2}-\lambda)}{2\pi i\Gamma(\frac{1}{2})}\exp(\pi i\lambda)(\frac{z}{2})^\lambda\int_{C2}(\zeta^2-1)^{\lambda-\frac{1}{2}}\exp(iz\zeta)d\zeta$$

where λ and z are complex numbers such that $z\neq 0$, and C_1, C_2 are curves in fig. A.

Since the integral in the third term of (4) is analytic function of λ, we may suppose $Re(\lambda)>0$. Taking Taylor's expansion form the integral is $\sum_{k=0}^\infty(iz)^k(k!)^{-1}\int_{C_1}\zeta^k(\zeta^2-1)^{\lambda-\frac{1}{2}}d\zeta$. Considering $(\zeta^2-1)^{\lambda-\frac{1}{2}}$ on its principal branch and closing C_1 to the segment with end points -1, $+1$

$$\int_{C_1}\zeta^k(\zeta^2-1)^{\lambda-\frac{1}{2}}d\zeta=\int_{-1}^1\left\{\exp(-i\pi(\lambda-\frac{1}{2}))-\exp(i\pi(\lambda-\frac{1}{2}))\right\}(1-\zeta^2)^{\lambda-\frac{1}{2}}\zeta^k d\zeta$$

$$=\begin{cases}-2i\sin(\lambda-\frac{1}{2})\pi\int_0^1(1-t)^{\lambda-\frac{1}{2}}t^{j-\frac{1}{2}}dt & (k=2j)\\ 0 & (k=2j+1).\end{cases}$$

Changing variable twice we have equal term to the last inteqral for $k=2j$ such that

$$2\int_0^{\frac{\pi}{2}}(\cos\theta)^{2j}(\sin\theta)^{2\lambda}d\theta=\frac{2}{\Gamma(\lambda+j+1)}\int_0^\infty e^{-r}r^{\lambda+j}dr\int_0^{\frac{\pi}{2}}(\cos\theta)^{2j}(\sin\theta)^{2\lambda}d\theta$$

$$=\frac{1}{\Gamma(\lambda+j+1)}\int_0^\infty e^{-s}s^{j-\frac{1}{2}}ds\int_0^\infty e^{-t}t^{\lambda-\frac{1}{2}}dt\quad \begin{pmatrix}s=r\cos^2\theta\\ t=r\sin^2\theta\end{pmatrix}$$

$$=\frac{\Gamma(j+\frac{1}{2})\Gamma(\lambda+\frac{1}{2})}{\Gamma(\lambda+j+1)}$$

Thus, from the relation $\Gamma(z)\Gamma(1-z)=\pi/\sin\pi z$ we have

$$\int_{C_1}\zeta^k(\zeta^2-1)^{\lambda-\frac{1}{2}}d\zeta=\begin{cases}\dfrac{2\pi i\Gamma(j+\frac{1}{2})}{\Gamma(\frac{1}{2}-\lambda)\Gamma(\lambda+j+1)} & (k=2j)\\ 0 & (k=2j+1).\end{cases}$$

Combining this formula with the above Taylor's expantion form, we know that the third term of (4) equal to the Bessel function $J_\lambda(z)$. At the integral in the third term of (4) close C_1 to the segment with end points -1 and $+1$, then we have

$$\int_{-1}^{1}\left\{\exp(-i\pi(\lambda-\frac{1}{2})-\exp(i\pi(\lambda-\frac{1}{2}))\right\}(1-\zeta^2)^{\lambda-\frac{1}{2}}\exp(iz\zeta)d\zeta$$
$$=2i\sin\pi(\frac{1}{2}-\lambda)\int_{-1}^{1}(1-t^2)^{\lambda-\frac{1}{2}}\exp(izt)dt.$$

Thus (4) is established. From the same argument on the curve C_2 as that on the curve C_1 (5) is derived.

Now we shall prove (1)−(3). For elements x and ξ in R_n, let ξ be fixed and we take a orthogonal transformation $y_j=\sum_{k=1}^{n} T_{jk}x_k$ with determinant $+1$ such that

$$T_{11}:T_{12}:\ldots:T_{1n}=\xi_1:\xi_2:\ldots:\xi_n.$$

Since $\sum_1^n x_k^2=\sum_1^n y_k^2$, $\sum_1^n \xi_k x_k=|\xi|\,.\,y_1$, writing x_k instead of y_k and taking polar-coordinate on R_{n-1} we have

$$\hat{f}(\xi_1,\xi_2\ldots,\xi_n)=\int_{R_1}\exp(-2\pi i|\xi|x_1)(\int_{R_{n-1}}\varphi(|x|)dx_2\ldots dx_n)dx_1$$
$$=\int_{-\infty}^{\infty}\exp(-2\pi i|\xi|x_1)(\int_{\Sigma}d\sigma\int_0^{\infty}\varphi(\sqrt{x_1^2+\rho^2})\rho^{n-2}d\rho)dx_1$$
$$=\omega_{n-1}\int_{-\infty}^{\infty}\int_0^{\infty}\varphi(\sqrt{x_1^2+\rho^2})\rho^{n-2}\exp(-2\pi i|\xi|x_1)d\rho dx_1$$

where Σ is the surface of unit sphere in R_{n-1} and $\omega_{n-1}=2\pi^{\frac{n-1}{2}}(\Gamma(\frac{n-1}{2}))^{-1}$

Again taking polar-coordinate $x_1=r\cos\theta$, $\rho=r\sin\theta$ for $0\leqq r<\infty$, $0\leqq\theta\leqq\pi$ and changing variable $-\cos\theta=t$, the last integral above is deduced to

$$\int_0^{\infty}\varphi(r)r^{n-1}(\int_{-1}^{1}(1-t^2)^{\frac{n-2}{2}-\frac{1}{2}}\exp(2\pi i|\xi|rt)dt)dr,$$

so that we have (1) from (4).

For the proof of (2) and (3) we use the relation ;

$$\frac{J_\lambda(z)}{z^\lambda}=-\frac{1}{z}\frac{d}{dz}(\frac{J_{\lambda-1}(z)}{z^{\lambda-1}})$$

which is obtained by differentiating $z^{1-\lambda}J_{\lambda-1}(z)$ in its form of series. If we substitute $z=t^{1/2}\rho$, ρ being fixed, and differentiate the form m times we have

$$\frac{J_\lambda(t^{1/2}\rho)}{t^{\lambda/2}}=(-\frac{2}{\rho})^m\frac{d^m}{dt^m}(\frac{J_{\lambda-m}(t^{1/2}\rho)}{t^{\frac{\lambda-m}{2}}}).$$

When we substitute $|\xi|^2=t$ in (1) we have (2) and (3) from this result.

§ 2. From the theorem in § 1 we can derive the Fourier transform of k_s such that

$$\text{(6)}\qquad \hat{k}_s(x)=\frac{2^{s/2}(2\pi)^{\frac{n+1}{2}}e^{-2\pi|x|}}{\Gamma(-\frac{s}{2})\Gamma(\frac{n+s+1}{2})}\int_0^{\infty}(t+\frac{t^2}{2})^{\frac{n+s-1}{2}}e^{-2\pi|x|t}dt \qquad \text{for } 0>s>-n-1.$$

Proof. First we prove the following formulas by the same consideration as in § 1.

$$\text{(7)}\qquad J_\lambda(z)=\frac{-1}{2\pi i}\int_c\frac{\Gamma(-\zeta)}{\Gamma(\lambda+\zeta+1)}(\frac{z}{2})^{\lambda+2\zeta}d\zeta$$

$$\text{(8)}\qquad \frac{1}{2\pi i}\int_C\Gamma(-\zeta)\Gamma(-\lambda-\zeta)(\frac{|z|}{2})^{\lambda+2\zeta}d\zeta$$
$$=\frac{\pi}{\sin\pi\lambda}\exp(\frac{\pi}{2}i\lambda)\{\exp(-\pi i\lambda)J_\lambda(i|z|)-J_{-\lambda}(i|z|)\}$$
$$=\frac{-2^{1-\lambda}\sqrt{\pi}}{\Gamma(\lambda+\frac{1}{2})}|z|^{-1/2}e^{-|z|}\int_0^{\infty}(2t+\frac{t^2}{|z|})^{\lambda-\frac{1}{2}}e^{-t}dt$$

where λ and $z(\neq0)$ are complex numbers and C is the curve in fig. A [6].

The integrand in the right term of (7) has poles with order 1 at $\zeta=0,1,2,\ldots$ and the residue which corresponds to $\zeta=m$ is equal to $(-1)^{m+1}(m!)^{-1}(\Gamma(\lambda+m+1))^{-1}(\frac{z}{2})^{\lambda+2m}$. Summing up these residues we have (7). Similarly the integrand in the first term of (8) has poles at $\zeta=0, 1, \ldots, -\lambda, -\lambda+1, \ldots$ and using the relation $\Gamma(z)\Gamma(1-z)=\pi/\sin\pi z$ the residues at $\zeta=m, -\lambda+m$ are equal to

$$\frac{\pi}{m!\Gamma(\lambda+m+1)\sin\pi\lambda}(\frac{|z|}{2})^{\lambda+2m}, \qquad \frac{-\pi}{m!\Gamma(-\lambda+m+1)\sin\pi\lambda}(\frac{|z|}{2})^{-\lambda+2m}.$$

By the condition of C, $-\lambda+m$ is also enclosed in C, so that the first term of (8) is equal to the second.

For the proof of second equality in (8) we use (4) and (5), substituting for integrals on the curve $C_1'\cup C_2'$ which is equivalent to the curve C_1 (see fig. B.). Now (4) is invariable for this substitution, meanwhile if we take the integrand in (5) on its principal branch

$$J_{-\lambda}(z)=\frac{\Gamma(\frac{1}{2}-\lambda)}{2\pi i\,\Gamma(\frac{1}{2})}\exp(\pi i\lambda)(\frac{z}{2})^{\lambda}(\int_{C_1'}-\exp(-2\pi i(\lambda-\frac{1}{2}))\int_{C_2'})(\zeta^2-1)^{\lambda-\frac{1}{2}}\exp(iz\zeta)d\zeta$$

$$=\frac{\Gamma(\frac{1}{2}-\lambda)(\frac{z}{2})^{\lambda}}{2\pi i\Gamma(\frac{1}{2})}(\exp(\pi i\lambda)\int_{C_1'}+\exp(-\pi i\lambda)\int_{C_2'})(\zeta^2-1)^{\lambda-\frac{1}{2}}\exp(iz\zeta)d\zeta,$$

so that

$$\pi\exp(\frac{\pi}{2}i\lambda)\Big\{\exp(-\pi i\lambda)J_{\lambda}(z)-J_{-\lambda}(z)\Big\}$$

$$=-\sin\pi\lambda\frac{\Gamma(\frac{1}{2}-\lambda)}{\Gamma(\frac{1}{2})}\exp(\frac{\pi}{2}i\lambda)(\frac{z}{2})^{\lambda}\int_{C_1'}(\zeta^2-1)^{\lambda-\frac{1}{2}}\exp(iz\zeta)d\zeta.$$

Changing variable such as $\zeta-1=z^{-1}it$ and closing the equivalent curve C in fig. B to the half line $[0, \infty)$, we have the following form ;

$$\frac{-\sin\pi\lambda\Gamma(\frac{1}{2}-\lambda)}{2^{\lambda}\Gamma(\frac{1}{2})}\exp(\pi i\lambda+iz)\, i^{1/2}z^{-1/2}\int_C(2t+\frac{it^2}{z})^{\lambda-\frac{1}{2}}\exp(-t)dt$$

$$=\frac{-\sin\pi\lambda\Gamma(\frac{1}{2}-\lambda)\exp(\pi i\lambda+iz)i^{1/2}z^{-1/2}}{2^{\lambda}\Gamma(\frac{1}{2})}\Big\{1-\exp(-2\pi i(\lambda+\frac{1}{2}))\Big\}\int_0^{\infty}(2t+\frac{it^2}{z})^{\lambda-\frac{1}{2}}e^{-t}dt$$

$$=\frac{-\sin\pi\lambda\; 2^{1-\lambda}\sqrt{\pi}}{\Gamma(\frac{1}{2}+\lambda)}i^{1/2}z^{-1/2}\exp(iz)\int_0^{\infty}(2t+\frac{it^2}{z})^{\lambda-\frac{1}{2}}e^{-t}dt.$$

By the analytic continuation this equality stands for $Re(z)=0$ and by the continuity of each term the second equality of (8) stands.

Now we can prove (6). Since $k_s\in L_1$ for $s<-n$, usdng (1), (7) and (8) and changing variable such as $\rho=\tan\theta$, we have

$$\hat{k}_s(x)=(2\pi)^{-n/2}|x|^{-n/2+1}\int_0^{\infty}(1+\frac{\rho^2}{4\pi^2})\rho^{n/2}J_{\frac{n-2}{2}}(|x|\rho)d\rho \qquad (x\in R_n)$$

$$=-\frac{(2\pi)^{n/2}}{2\pi i}|x|^{1-n/2}(\frac{|x|}{2})^{n/2-1}\int_C\frac{\Gamma(-\zeta)(\frac{|x|}{2})^{2\zeta}}{\Gamma(\frac{n}{2}+\zeta)}(2\pi)^{2\zeta}\int_0^{\infty}(1+\rho^2)^{S/2}\rho^{n-1+2\zeta}d\rho d\zeta$$

$$=-\frac{\pi^{n/2-1}}{i}\int_C\frac{\Gamma(-\zeta)(\pi|x|)^{2\zeta}}{\Gamma(\frac{n}{2}+\zeta)}\int_0^{\frac{\pi}{2}}(\cos\theta)^{-S-n-2\zeta-1}(\sin\theta)^{n+2\zeta-1}d\theta$$

$$= \frac{-(2\pi)^{\frac{n-s}{2}}|x|^{-\frac{n+s}{2}}}{2^{\frac{n-s}{2}+1}\pi^{n/2+1}\, i\Gamma(\frac{-s}{2})} \int_C \Gamma(-\zeta)\Gamma(-\frac{n+s}{2}-\zeta)(\pi|x|)^{\frac{n+s}{2}+2\zeta}\, d\zeta \qquad \text{(see p. 2)}$$

$$= \frac{2(2|x|)^{-\frac{n+s+1}{2}}\, e^{-2\pi|x|}}{\pi^{\frac{s}{2}}\Gamma(\frac{-s}{2})\Gamma(\frac{n+s+1}{2})} \int_0^\infty (2t+\frac{t^2}{2\pi|x|})^{\frac{n+s-1}{2}} e^{-t}\, dt$$

$$= \frac{2^{s/2}(2\pi)^{\frac{n+1}{2}}\, e^{-2\pi|x|}}{\Gamma(\frac{-s}{2})\Gamma(\frac{n+s+1}{2})} \int_0^\infty (t+\frac{t^2}{2})^{\frac{n+s-1}{2}} e^{-2\pi|x|t}\, dt.$$

Thus we have (6) for $s<-n$. If $\hat{k}_s$ is defined by (6) it is integrable for all s such that $0>s>-n-1$, so that by applying the middle form in (8) and Bochner's theorem to $\hat{k}_s$ we have $\hat{\hat{k}}_s=\breve{k}_s$.

This completes the argument.

§ 3. Let u be the element of $\mathscr{H}_{(s)}$ and let ψ be the test function such that

$$\int_{R_n}\psi(x)dx=1, \qquad \psi(x)\geqq 0, \qquad \operatorname{supp}\psi=\{x\,;\,|x|\leqq 1\}.$$

When $\psi_\varepsilon(x)=\varepsilon^{-n}\psi(\frac{x}{\varepsilon})$, it is known that $u*\psi_\varepsilon$ is infinitely differentiable function and $\|u*\psi_\varepsilon*\hat{k}_s\|_2=\|\widehat{(u*\psi_\varepsilon)}\cdot\breve{k}_s\|_2=\|u*\psi_\varepsilon\|_{(s)}$ if the one is finite by Parseval's equation.

Now we may show that

(9) $\quad \|u*\psi_\varepsilon\|_{(s)}\leqq A\|u*\psi_\varepsilon\|_2 \qquad$ for $0>s>-n-1$.

Proof of (9). Taking polar-coordinate in (6) and using Minkowski's inequality we have

$$\text{(10)}\qquad \|(u*\psi_\varepsilon)*\hat{k}_s\|_2=\|\int_0^\infty(t+\frac{t^2}{2})^{\frac{n+s-1}{2}}e^{-2\pi|y|t}dt\int_{R_n}A_{ns}u*\psi_\varepsilon(x-y)e^{-2\pi|y|}dy\|_2$$

$$=A\|\int_0^\infty(t+\frac{t^2}{2})^{\frac{n+s-1}{2}}dt\int_\Sigma d\sigma\int_0^\infty u*\psi_\varepsilon(x-\rho\sigma)e^{-2\pi\rho(1+t)}\rho^{n-1}d\rho\|_2$$

$$\leqq A\|u*\psi_\varepsilon\|_2\frac{2^n\pi^{\frac{n-1}{2}}\Gamma(\frac{n+1}{2})}{\Gamma(n)}\int_0^\infty(t+\frac{t^2}{2})^{\frac{n+s-1}{2}}dt\int_0^\infty e^{-2\pi\rho(1+t)}\rho^{n-1}d\rho$$

$$\leqq\frac{A\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}\|u*\psi_\varepsilon\|_2\int_0^\infty(t+\frac{t^2}{2})^{\frac{n+s-1}{2}}(1+t)^{-n}dt.$$

If $0>s>-n-1$, the last integral is finite then we have proved (9).

Remark. The inner integral in L_2 norm on the right hand side of (10) is equal to $u*\psi_\varepsilon*\hat{P}(x,1+t)$.

figure A

(ζ-plane)

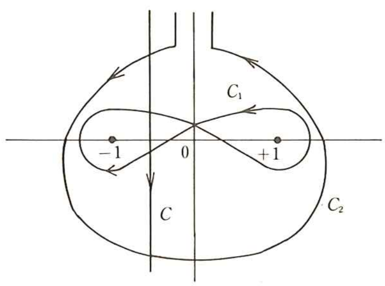
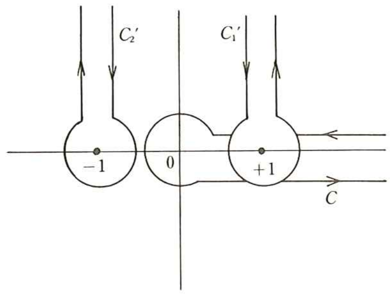


figure B

(t−plane)



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(*) This is the set of all (x, y) such that $x \in R_n$, $y > 0$.

(**) This is the set of all temperate distributions.

(***) Convolution of u with test function.